

Relativistic SUSY QM as Deformed SUSY QM

M.Arik, T.Rador,

Department of Physics, Bogazici University
Bebek 80815,
Istanbul Turkey

R.Mir-Kasimov

Bogoliubov Laboratory of Theoretical Physics
Joint Institute for Nuclear Research
Dubna Moscow region
141980 Russia

February 7, 2008

Abstract

The relativistic finite-difference SUSY Quantum Mechanics (QM) is developed. We show that it is connected in a natural way with the q-deformed SUSY Quantum Mechanics. Simple examples are given.

The relativistic QM is based on Snyder's idea of noncommutative space-time coordinates and the concept of relativistic configurational space. We refer the reader to the review papers [1-5], and the references therein. The natural connection of RQM with q-deformations is discussed in the articles [5, 6].

The one-dimensional relativistic Schrödinger equation has the form

$$(h - e) \psi(x) = (h_0 + V(x) - e) \psi(x) = 0, \quad (1)$$

where

$$h_0 = 2mc^2 \sinh^2 \frac{i\hbar}{2mc} \frac{d}{dx} = \hat{k}^2 \rightarrow -\frac{\hbar^2}{2m} \frac{d^2}{dx^2} \text{ in non-relativistic limit} \quad (2)$$

$$\hat{k} = -2mc \sinh \frac{i\hbar}{2mc} \frac{d}{dx}, \quad k = 2mc \sinh \frac{\chi}{2}, \quad e = \frac{k^2}{2}$$

We stress that the relativistic Schrödinger equation (1) is the finite-difference equation with a step equal to the Compton wave length $\frac{\hbar}{mc}$ of the particle. Usually, we shall use the unit system in which $\hbar = c = m = 1$. It is clear that the factorization method, that plays the key role in SUSY QM, must be modified in the finite-difference context [5]. Let us introduce a couple of ladder operators

$$A^\pm = \pm i\sqrt{2} \cdot \alpha(x) \cdot e^{\pm\rho(x)} \sinh \frac{i}{2} \frac{d}{dx} \cdot e^{\mp\rho(x)} \quad (3)$$

or

$$A^{\pm} = -i\sqrt{2} \cdot \alpha(x) \cdot e^{\pm\xi(x)} \left[\sinh \rho_{\frac{s}{2}}(x) \cosh \frac{i}{2} \frac{d}{dx} \mp \cosh \rho_{\frac{s}{2}}(x) \sinh \frac{i}{2} \frac{d}{dx} \right] \quad (4)$$

where

$$\rho_{\frac{s}{2}}(x) = \sinh \frac{i}{2} \frac{d}{dx} \rho(x) \quad \rho_{\frac{c}{2}}(x) = \cosh \frac{i}{2} \frac{d}{dx} \rho(x) \quad (5)$$

$$\xi(x) = \rho(x) - \rho_{\frac{c}{2}}(x) \quad (6)$$

and $\rho(x)$ is the logarithm of the ground state wave function of eq. (1)

$$\psi_0(x) = e^{-\rho(x)} \quad (7)$$

In the nonrelativistic limit the operators A^{\pm} turn into the usual ladder operators (cf. [5, 6]). In addition to the finite-difference character of the operators (3, 4) there are two factors $\alpha(x)$ and $e^{\pm\xi(x)}$ that don't appear in the nonrelativistic case and whose presence enforces the difference with the nonrelativistic ladder operators. The factor $\alpha(x)$ is connected with a natural lattice variable and is also expressed in terms of $\rho(x)$ (see [5, 9]). Factors $e^{\pm\xi(x)}$ are connected with deformations. Some quantity (deformation)

$$q(x) = e^{a(x)} \quad (8)$$

must be introduced to cancel $e^{\pm\xi(x)}$. In the nonrelativistic case $\xi(x) \rightarrow 1$ and there is no deformation : $q(x) \rightarrow 1$. Let us consider the $q(x)$ -mutator

$$\begin{aligned} [A^-, A^+]_{q(x)} &= A^- \cdot q(x) \cdot A^+ - A^+ \cdot q^{-1}(x) \cdot A^- = \\ &= \frac{\alpha(x)}{2} \left\{ \begin{aligned} &e^{\frac{i}{2} \frac{d}{dx}} \sinh Z(x) \alpha(x) e^{\frac{i}{2} \frac{d}{dx}} + e^{-\frac{i}{2} \frac{d}{dx}} \sinh Z(x) \alpha(x) e^{-\frac{i}{2} \frac{d}{dx}} - \\ &- e^{\frac{i}{2} \frac{d}{dx}} \sinh \left(Z(x) + 2\rho_{\frac{s}{2}}(x) \right) \alpha(x) e^{-\frac{i}{2} \frac{d}{dx}} - \\ &- e^{-\frac{i}{2} \frac{d}{dx}} \sinh \left(Z(x) - 2\rho_{\frac{s}{2}}(x) \right) \alpha(x) e^{\frac{i}{2} \frac{d}{dx}} \end{aligned} \right\} \quad (9) \end{aligned}$$

where

$$Z(x) = 2\xi(x) + a(x) \quad (10)$$

Let us recall that the commutator of the nonrelativistic ladder operators a^{\pm} does not contain the differentiation operators

$$[a^-, a^+] = \frac{d^2 \rho(x)}{dx^2} \quad (11)$$

By analogy with (9) we shall require that there is no finite-difference derivatives $e^{\pm \frac{i}{2} \frac{d}{dx}}$ in the r.h.s. of (9). The simplest way to achieve this is to put

$$Z(x) = 0 \quad (12)$$

The last equation gives the relation connecting $\rho(x)$ and $a(x)$ (or $q(x)$):

$$a(x) = -2\xi(x) \quad q(x) = e^{-2\xi(x)} \quad (13)$$

We have

$$\begin{aligned} [A^-, A^+]_{q(x)} &= -2\alpha(x) \sinh \frac{i}{2} \frac{d}{dx} \left[\alpha(x) \sinh 2\rho_{\frac{s}{2}}(x) \right] \rightarrow \\ &\rightarrow \frac{d^2 \rho(x)}{dx^2} \end{aligned} \quad (14)$$

Now let us write down the basic relations of relativistic SUSY QM, i.e., the relativistic quantum mechanical system whose Hamiltonian is constructed of anticommuting charges Q [10]:

$$\hat{H} = \frac{1}{2} \cdot \{Q, Q^\dagger\}_{q(x)} = \frac{1}{2} \cdot (Q \cdot q(x)^{-1} \cdot Q^\dagger + Q^\dagger \cdot q(x) \cdot Q) \quad (15)$$

$$\{Q, Q\} = \{Q^\dagger, Q^\dagger\} = 0 \quad (16)$$

As in the nonrelativistic, case the supersymmetry property of Hamiltonian \hat{H}

$$[\hat{H}, Q] = [\hat{H}, Q^\dagger] = 0 \quad (17)$$

is provided by nilpotency of the charge operators (16). The Hamiltonian \hat{H} contains coordinates that are quantized by $q(x)$ -mutators and anticommutators. They are mixed by deformed supersymmetry transformations. The explicit realization of Q and Q^\dagger is

$$Q = i\sqrt{2} \cdot A^+ \cdot \hat{\psi}^\dagger, \quad Q^\dagger = -i\sqrt{2} \cdot A^- \cdot \hat{\psi} \quad (18)$$

In the simplest case the bosonic degrees of freedom represented by the ladder operators A^\pm are described by the momentum operator (2) and the position operator x with the commutation relation

$$[x, \hat{k}] = i \cosh \frac{i}{2} \frac{d}{dx} \quad (19)$$

whereas $\hat{\psi}^\dagger$ and $\hat{\psi}$ are Fermi degrees of freedom with the corresponding anticommutation relations:

$$\left\{ \hat{\psi}^\dagger, \hat{\psi} \right\} = 1, \quad \left\{ \hat{\psi}, \hat{\psi} \right\} = \left\{ \hat{\psi}^\dagger, \hat{\psi}^\dagger \right\} = 0 \quad (20)$$

This yields (16) and

$$\hat{H} = H - \frac{1}{2} \cdot \left[\hat{\psi}^\dagger, \hat{\psi} \right] \cdot \Delta V(x). \quad (21)$$

We introduce the operator

$$\begin{aligned} H &= \frac{1}{2} \cdot \{A^-, A^+\}_{q(x)} = \frac{1}{2} \cdot (A^- \cdot q(x) \cdot A^+ + A^+ \cdot q^{-1}(x) \cdot A^-) = \\ &= H_0 + \alpha(x) \alpha_{\frac{s}{2}}(x) - \alpha(x) \cdot \cosh \frac{i}{2} \frac{d}{dx} \left(\alpha(x) \cdot \cosh 2\rho_{\frac{s}{2}}(x) \right) \rightarrow \\ &\rightarrow -\frac{1}{2} \frac{d^2}{dx^2} + \frac{1}{2} (\rho'(x))^2, \end{aligned} \quad (22)$$

where the operator

$$H_0 = 2 \left[\alpha(x) \cdot \sinh \frac{i}{2} \frac{d}{dx} \right]^2 = \frac{\hat{P}^2}{2} \rightarrow -\frac{1}{2} \frac{d^2}{dx^2} \quad (23)$$

plays the role of free hamiltonian with the momentum operator

$$\hat{P} = -2\alpha(x) \cdot \sinh \frac{i}{2} \frac{d}{dx} \quad (24)$$

modified by the interaction

$$H_+ = A^+ \cdot q^{-1}(x) \cdot A^- = H - \Delta V(x) \quad (25)$$

$$H_- = A^- \cdot q(x) \cdot A^+ = H + \Delta V(x)$$

$$\Delta V(x) = -\alpha(x) \sinh \frac{i}{2} \frac{d}{dx} \left[\alpha(x) \sinh 2\rho_{\frac{s}{2}}(x) \right] \quad (26)$$

In the (2×2) -representation

$$\hat{\psi}^\dagger = \sigma_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad \hat{\psi} = \sigma_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad (27)$$

$$\left[\hat{\psi}^\dagger, \hat{\psi} \right] = -\sigma_3 \quad (28)$$

we find from (21)

$$\begin{aligned} \hat{H} &= H + \frac{1}{2} \cdot \Delta V(x) \cdot \sigma_3 = \\ &= \begin{pmatrix} H_- & 0 \\ 0 & H_+ \end{pmatrix} = \begin{pmatrix} A^- \cdot q(x) \cdot A^+ & 0 \\ 0 & A^+ \cdot q^{-1}(x) \cdot A^- \end{pmatrix} \end{aligned} \quad (29)$$

EXAMPLES

1) *RELATIVISTIC OSCILLATOR* (q-oscillator) [5]- [8].

In this case, we have [5, 6]:

$$\rho(x) = \frac{m\omega x^2}{2\hbar} \quad (30)$$

The deformation parameter is a constant and we come to the q -oscillator with

$$q(x) = \text{const} = q = e^{-\frac{\omega\hbar}{4mc^2}} \quad (31)$$

and

$$\alpha(x) = \frac{1}{\cos \frac{\omega x}{2c}}. \quad (32)$$

The finite-difference ladder operators have the form

$$A^\pm = \pm i\sqrt{2} \cdot e^{\pm \frac{\omega}{8}} \cdot \left(\sinh \frac{i}{2} \frac{d}{dx} \mp i \tan \frac{\omega x}{2} \cdot \cosh \frac{i}{2} \frac{d}{dx} \right). \quad (33)$$

SUSY Hamiltonian (29) becomes

$$\begin{aligned} \hat{H} &= \begin{pmatrix} e^{-\frac{\omega}{4}} A^- A^+ & 0 \\ 0 & e^{\frac{\omega}{4}} A^+ A^- \end{pmatrix} = \begin{pmatrix} h + e_0 & 0 \\ 0 & h - e_0 \end{pmatrix} \rightarrow \\ &\rightarrow \begin{pmatrix} -\frac{1}{2} \frac{d^2}{dx^2} + \frac{\omega^2 x^2}{2} + \frac{\omega}{2} & 0 \\ 0 & -\frac{1}{2} \frac{d^2}{dx^2} + \frac{\omega^2 x^2}{2} - \frac{\omega}{2} \end{pmatrix} \end{aligned} \quad (34)$$

where

$$e_0 = 2 \sinh \frac{\omega}{4} \rightarrow \frac{\omega}{2} \quad (35)$$

and

$$h = \{A^-, A^+\}_q = 2 \left\{ \left(\frac{1}{\cos \frac{\omega x}{2}} \cdot \cosh \frac{i}{2} \frac{d}{dx} \right)^2 - \cosh \frac{\omega}{4} \right\} = \quad (36)$$

$$= \frac{\hat{P}}{2} + V(x) \\ \hat{P} = -\frac{2}{\cos \frac{\omega x}{2}} \cdot \sinh \frac{i}{2} \frac{d}{dx} \rightarrow -i \frac{d}{dx} \quad (37)$$

The relativistic oscillator potential is

$$V(x) = \frac{\cosh \frac{\omega}{4} \cdot \left[\sin^2 \frac{\omega x}{2} - \sinh^2 \frac{\omega}{4} \right]}{\left[\cos^2 \frac{\omega x}{2} + \sinh^2 \frac{\omega}{4} \right]} \rightarrow \frac{\omega^2 x^2}{2} \quad (38)$$

The spectrum is

$$\left(\begin{array}{cc} e_n^- = 2 \left(e^{\frac{2n+1}{4}\omega} - e^{-\frac{\omega}{4}} \right) & 0 \\ 0 & e_n^+ = 2 \left(e^{\frac{2n+1}{4}\omega} - e^{\frac{\omega}{4}} \right) \end{array} \right) \rightarrow \quad (39) \\ \rightarrow \left(\begin{array}{cc} \left(n + \frac{1}{2} \right) \omega + \frac{\omega}{2} & 0 \\ 0 & \left(n + \frac{1}{2} \right) \omega - \frac{\omega}{2} \end{array} \right)$$

Thus, we have two q -oscillators with zero point of energy shifted by $\pm e_0$. In other words, the energies of q -supersymmetric partners are connected by

$$e_{n+1}^- = q^{-2} \cdot e_n^+ \quad (40)$$

2) THE RADIAL PART OF THREE - DIMENSIONAL RELATIVISTIC SCHRÖDINGER EQUATION [2, 3, 4, 5]:

This equation

$$H_l s_l(r, \chi) = \left\{ 2 \sinh^2 \frac{i}{2} \frac{d}{dr} + \frac{l(l+1)}{2r(r+i)} e^{i \frac{d}{dr}} \right\} s_l(r, \chi) = \quad (41) \\ = (\cosh \chi - 1) s_l(r, \chi)$$

can be considered as one-dimensional with the potential $\frac{l(l+1)}{2r(r+i)} e^{i \frac{d}{dr}}$. Solutions of this equation, i.e., the free relativistic radial waves have the form

$$s_l(r, \chi) = \sqrt{\frac{\pi \sinh \chi}{2}} \cdot (-i)^{l+1} \cdot \frac{\Gamma(ir + l + 1)}{\Gamma(ir)} \cdot P_{ir - \frac{1}{2}}^{-(l + \frac{1}{2})}(\cosh \chi) \quad (42)$$

In the nonrelativistic limit, these functions turn into free solutions of the Schrödinger equation

$$s_l(r, \chi) \rightarrow s_l(pr) = \sqrt{\frac{\pi r p}{2}} \cdot J_{l + \frac{1}{2}}(pr) \quad (43)$$

In this case the relativistic finite-difference ladder operators have the form

$$\lambda^\pm = \pm \frac{i}{\sqrt{2}} \left[\frac{ir \mp (l+1)}{ir} - e^{i \frac{d}{dr}} \right] \rightarrow a^\pm = \mp \left[\frac{d}{dr} \pm \frac{l+1}{r} \right] \quad (44)$$

Hence,

$$H_+ = H_l = \lambda^+ \cdot e^{i\frac{d}{dr}} \cdot \lambda^- \quad H_- = H_{l+1} = \lambda^- \cdot e^{i\frac{d}{dr}} \cdot \lambda^+ \quad (45)$$

In contrast with the nonrelativistic case, the rising and lowering operators Λ^\pm , which shift the value of the angular momentum

$$\Lambda^+ s_{l+1}(r, \chi) = s_l(r, \chi) \quad \Lambda^- s_l(r, \chi) = s_{l+1}(r, \chi) \quad (46)$$

and the ladder operators (44) factorizing the Hamiltonian are different:

$$\begin{aligned} \Lambda^+ &= \frac{i}{\sinh \chi} \cdot \left[\cosh \chi - \frac{ir-l-2}{ir-1} \cdot e^{i\frac{d}{dr}} \right] \rightarrow a^+ \\ \Lambda^- &= -\frac{i}{\sinh \chi} \cdot \left[\cosh \chi - \frac{ir+l}{ir-1} \cdot e^{i\frac{d}{dr}} \right] \rightarrow a^- \end{aligned} \quad (47)$$

Let us consider the identity

$$[-H_{l+1} + H_l] \cdot H_l - [H_{l+1} - H_l] \cdot H_l \equiv 0 \quad (48)$$

Using the relation

$$\lambda^- e^{i\frac{d}{dr}} - e^{i\frac{d}{dr}} \lambda^- = -\frac{i}{\sqrt{2}} \cdot [H_{l+1} - H_l] \quad (49)$$

we have

$$[-H_{l+1} + H_l] \cdot H_l - i\sqrt{2} \cdot \left[\lambda^- e^{i\frac{d}{dr}} - e^{i\frac{d}{dr}} \lambda^- \right] \cdot H_l = 0 \quad (50)$$

After acting on $s_l(r, \chi)$ and taking into account (41) and the relation

$$\Lambda^- = \frac{i}{\sinh \chi} \cdot \left[-(\cosh \chi - 1) + i\sqrt{2} \cdot e^{i\frac{d}{dr}} \cdot \lambda^- \right], \quad (51)$$

we come to the formula

$$H_{l+1} \cdot \Lambda^- s_l(r, \chi) = \Lambda^- \cdot H_l s_l(r, \chi) \quad (52)$$

which allows us to consider relativistic l and $l+1$ states as deformed supersymmetric partner states. If $s_l(r, \chi)$ is the eigenstate of H_l , then $(\Lambda^- s_l(r, \chi))$ is the eigenstate of H_{l+1} with the same eigenvalue

$$\begin{aligned} H_l s_l(r, \chi) &= (\cosh \chi - 1) \cdot s_l(r, \chi) \rightarrow \\ &\rightarrow H_{l+1} \cdot (\Lambda^- s_l(r, \chi)) = (\cosh \chi - 1) \cdot (\Lambda^- s_l(r, \chi)) \end{aligned} \quad (53)$$

The nonrelativistic (undeformed) analog of (52) is the relation

$$H_{l+1} \cdot a^- s_l(pr) = a^- \cdot H_l s_l(pr) \quad (54)$$

References

- [1] V.G.Kadyshevsky, in book *Problems of theoretical Physics*, dedicated to the memory of I.E.Tamm *Nauka Publishers*, Moscow (1972) 52.
- [2] V.G.Kadyshevsky, R.M.Mir-Kasimov and N.B.Skachkov, *Nuovo Cimento* **55 A** (1968) 233.
- [3] V.G.Kadyshevsky, R.M.Mir-Kasimov and N.B.Skachkov, *Physics of Elementary Particles and Atomic Nucleus* **2**, N3, (1972) 635
- [4] I.V.Amirkhanov, G.V.Grusha and R.M.Mir-Kasimov, *Physics of Elementary Particles and Atomic Nucleus* **12**, N3, (1981) 651
- [5] R.M.Mir-Kasimov, **Preprint** SISSA 197/94/EP, (1994)
- [6] R.M.Mir-Kasimov, **Preprint** Centre de Recherches Mathematiques, Universite de Montreal, CRM-2186, (1994); E.D.Kagramanov, R.M.Mir-Kasimov and Sh.M.Nagiyev, *J.Math.Phys.*, **31**, (1990) 1733; R.M.Mir-Kasimov, *J. Phys. A* **24** (1991) 4283
- [7] A.J.Macfarlane, *J.Phys.A***22** (1989) 4581; L.C.Biedenharn ibid **22** (1989) L873.
- [8] M.Arik, Ph.D. Thesis, University of Pittsburgh (1974) unpublished; M.Arik, D.D.Coon, *J.Math.Phys.* 17 (1975) 524; M.Arik, M.Mungan, *Phys.Lett.B* **282** (1992) 101.
- [9] A.F.Nikiforov, S.K.Suslov, V.B.Uvarov *Classical Orthogonal Polynomials of a Discrete Variable* Springer-Verlag, Berlin-Heidelberg, (1991); G.Gasper and M.Rahman *Basic Hypergeometric Series*, Cambridge University Press (1990)
- [10] F.Cooper, A.Khare, U.Sukhatme, *Phys.Rep.***251**, N 5 & 6, (1995); F.Schwabl *Quantum Mechanics*, Springer-Verlag (1992)
- [11] A.T.Filippov, A.P.Isaev, *Mod.Phys.Lett.*,**A4** , (1989), 2167; A.P.Isaev and R.P.Malik, *Phys.Lett.B***280**, (1992), 219; V.I.Manko, G.Marmo, S.Solimeno and F.Zaccaria, *Phys.Lett.A* **176**,(1993) 173; L.A.Slepchenko,*Theor. and Math.Phys.***78**(1989) 211 ; V.Spiridonov, in the Proc. of Workshop on Harmonic Oscillators (College Park, USA, 25-28 March 1992). Eds. D.Han, Y.-S.Kim, and W.W.Zachary, NASA Conf. Publ. 3197, 199, pp. 93-108; A.N.Sissakian, V.M.Ter-Antonyan, G.S.Pogosyan, I.V.Lutsenko, *Phys.Lett A***143** (1990) 247; M.Chaichian, P.P.Kulish and J.Lukierski, *Phys.Lett* **B237** (1990) 401;